Inference for the Wiener process with random initiation time

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- Degradation models vs. Lifetime models? highly reliable components, use of complex preventive maintenance policies, etc.
- Current models: component degradation initiated when put in service!
- **Need of some new models:** models with an initiation period (deterministic or random)
  See Guo *et al.* (13), Nelson (10)

Degradation model

Degradation model with random initiation period $(X(t))_{t\geq 0}$:

$$X(t) = [\mu(t-S) + \sigma B(t-S)]\,\mathbb{I}_{t\geq S}$$

where

- $t = 0$ is the instant where the component is put in service
- $(B(t))_{t\geq 0}$ is a standard Brownian motion
- S is an absolutely continuous and positive random variable, independent of $(B(t))_{t\geq 0}$

Time-to-failure

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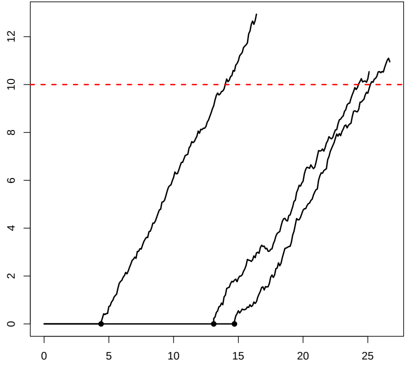
Special case: S exponentially distributed, see Schwarz (01, 02) with an application in psychology

Simulation of three sample paths:
black circles = degradation initiations
red dash line = critical level.



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Model assumptions? Parametric model for the distribution of S, with unknown parameter $\theta \in \Theta \subseteq \mathbb{R}^p$

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   Information on θ (interval-censoring), μ and σ^2

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- $\mathcal{N}_{2+}$: set of individuals with exactly at least two degradation measures:

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- Random vector $\underline{\mathcal{K}} = (\mathcal{K}_r)_{r \in \mathbb{N}^*}$ such that, for $r \in \mathbb{N}^*$,

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- Random number Q_n of non-null increments: if Q_n non empty set,

$$Q_n = \sum_{i\in\mathcal{N}_{2+}} (m - R_i) = \sum_{j=1}^{m-1} (m-j) K_j$$

  taking values in $\{1, \ldots, (m-1)n\}$

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3. $\mathbb{E}[Q_n^{-1}|Q_n > 0] \xrightarrow[n\to\infty]{} 0$

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- Maximum likelihood estimator:

$$\widehat{\theta}_n = \text{argmax}_{\theta\in\Theta}\ell(\theta|\text{data}).$$

No closed-form expression in general

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- Closed expression for the Fisher information

Example: exponential distribution

- Closed expression for the MLE:

$$\widehat{\lambda}_n = \frac{1}{\delta} \log \left(\frac{N_0 \tau + \delta \sum_{r=1}^{m} r K_r}{N_0 \tau + \delta \sum_{r=1}^{m} (r-1) K_r} \right)$$

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Remark: $\rho^2 \xrightarrow[\delta \to 0]{} \dfrac{\lambda^2}{1 - e^{-\lambda\tau}}$

Estimation of μ and σ^2 (1/2)

- Natural estimator of μ:

$$\widehat{\mu}_n = \frac{\sum_{i \in \mathcal{N}_{2+}} \sum_{j=1}^{m-R_i} \Delta X_{i,j}}{\delta \sum_{i \in \mathcal{N}_{2+}} (m - R_i)} = \frac{1}{\delta Q_n} \sum_{h=1}^{Q_n} Z_h,$$

where $Z_1, \ldots, Z_{Q_n}$ are the increments between two non-null degradation measures: random number of iid Gaussian random variables with mean $\mu\delta$ and variance $\sigma^2\delta$

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- Natural estimator of σ^2:

$$\widehat{\sigma}_n^2 = \frac{1}{\delta(Q_n - 1)} \sum_{h=1}^{Q_n} (Z_h - \delta\widehat{\mu}_n)^2.$$

Estimation of μ and σ^2 (2/2)

Proposition

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2. $\widehat{\sigma}_n^2$ *is asymptotically normal:*

$$\sqrt{Q_n}\left(\widehat{\sigma}_n^2 - \sigma^2\right) \xrightarrow[n\to\infty]{d} N\left(0, 2\sigma^4\right)$$

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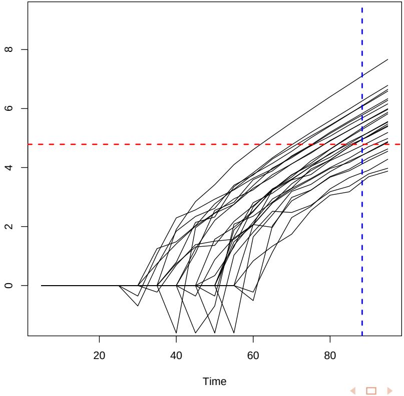
$$I(\theta)^{-1}\left(\int_0^\infty \partial_\theta \overline{F}_S(u;\theta)\mathrm{d}u\right)^2 + \frac{c^2\sigma^2}{\mu^4\tau\alpha(m,\tau)}$$

Guo *et al.* data

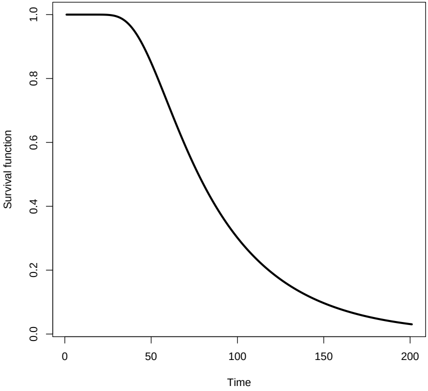
Black lines: observed degradation paths
Red dashed line: critical level
Blue dashed line: MTTF estimation



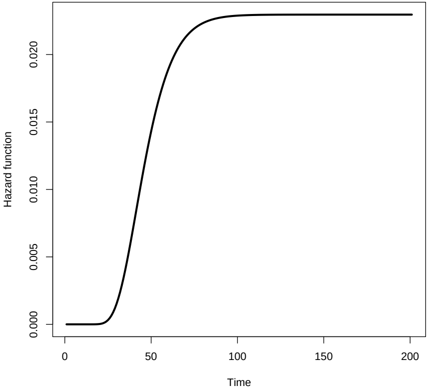
Fitted parameters

Parameter	Estimation	95% confidence interval
λ	0.023	[0.013,0.032]
μ	0.108	[0.097,0.119]
σ^2	0.041	[0.033,0.048]
MTTF	88.332	[69.438,107.227]

Estimated survival function



Estimated hazard function



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